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Processes of arbitrary order in quantum electrodynamics with a pair-creating external field

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Abstract. Dyson's perturbation theory analogue for quantum electrodynamical processes with arbitrary initial and final states in an external field creating pairs has been discussed. The interaction with the field is taken into account exactly. The possibility of using Feynman diagrams, together with modified correspondence rules, for the representation of the above mentioned processes has been demonstrated.

1. Introduction

The study of electrodynamical processes in an external field requires the exact evaluation of the field to all orders of the perturbation expansion. There are a number of approaches to the solution of this problem (Feynman 1949, Schwinger 1954, Furry 1951). However, a consistent treatment of perturbation theory and diagrammatic techniques for arbitrary processes has been carried out only for fields which do not create pairs.

The difficulties associated with pair creation manifest themselves most clearly, for instance, in the Furry (1951) approach: here the creation and annihilation operators for particles and antiparticles should be formed using solutions of the Dirac equation in the external field. For pair-creating fields, however, no solutions exist which could be attributed to particles or antiparticles for every time. These difficulties do not seem to be of major importance and should be thought of as evidence for the fact that no consistent one-particle interpretation of the Dirac equation is possible for external fields of arbitrary form. Nevertheless, exact solutions and Green functions of the Dirac equation undoubtedly contain all the information necessary for summing up the perturbation series with respect to the external field. This has been shown (e.g. by Feynman 1949, Schwinger 1954, Nikishov 1972, Grib *et al* 1972, Bagrov *et al* 1975) for processes of zeroth order with respect to the electron–photon interaction and for fields which cease to create pairs when $t \to \pm\infty$.

In the present paper we will consider an analogue of the Dyson perturbation theory for processes with arbitrary initial and final states for quantum electrodynamics in an external field which creates pairs. The interaction with the external field will be evaluated exactly. It will be shown that Feynman diagrams, with modified correspondence rules, may be used for the description of these processes.

2. General formulae for the transition probability amplitudes

Consider the quantum electrodynamical Hamiltonian in an external electromagnetic field. The sum of the Hamiltonians of the electron–positron field in an external electromagnetic field with potentials $\tilde{\mathscr{A}}$ and the free electromagnetic field $\mathscr{H}_{\rm rad}$ is chosen as the zeroth Hamiltonian $\mathscr{H}_0(t)$:

$$\mathscr{H}(t)=\mathscr{H}_0(t)+\int j^\mu(\boldsymbol{x})\mathscr{A}_\mu(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x},\qquad j^\mu(\boldsymbol{x})=\tfrac{1}{2}e[\bar\psi(\boldsymbol{x})\gamma^\mu,\psi(\boldsymbol{x})]_-,\tag{1}$$

$$\mathscr{H}_0(t)=\int :\bar\psi(\boldsymbol{x})(-\mathrm{i}\boldsymbol{\gamma}\nabla+e\hat{\mathscr{A}}(x)+m)\psi(\boldsymbol{x}):\,\mathrm{d}\boldsymbol{x}+\mathscr{H}_{\rm rad},\qquad \hat{\mathscr{A}}(x)=\tilde{\mathscr{A}}_\mu(x)\gamma^\mu.$$

Now we define the evolution operator of the electron–positron field in the external electromagnetic field:

$$\left(\mathrm{i}\frac{\partial}{\partial t}-\mathscr{H}_0(t)\right)U_0(t,t')=0,\qquad U_0(t,t')|_{t=t'}=1,$$

and construct, with its aid, the field operators in the 'interaction' picture:

$$\begin{aligned}\mathscr{A}_\mu(x)&=U_0^{-1}(t,t_1)\mathscr{A}_\mu(\boldsymbol{x})U_0(t,t_1), &\qquad \Box\mathscr{A}_\mu(x)&=0,\\ \psi(x)&=U_0^{-1}(t,t_1)\psi(\boldsymbol{x})U_0(t,t_1), &\qquad (\mathrm{i}\hat\partial-e\hat{\mathscr{A}}(x)-m)\psi(x)&=0.\end{aligned}\tag{2}$$

Then provided the operator $U_0(t,t')$ is known the total evolution operator $U(t,t')$ for the Hamiltonian (1) may be presented in a form for which the expansion in powers of the charge does not demand any expansion in powers of the external field:

$$U(t_2,t_1)=U_0(t_2,t_1)S(t_2,t_1),\qquad S(t_2,t_1)=T\exp\left(-\mathrm{i}\int_{t_1}^{t_2}j^\mu(x)\mathscr{A}_\mu(x)\,\mathrm{d}x\right).\tag{3}$$

When considering processes in an external field it is necessary to choose the initial and final states at the moments of time t_1 and t_2, respectively. In accordance with quantum mechanics one may, in principle, choose an arbitrary state at one given moment in time and consider the probability of transition to another arbitrary state at another given moment in time. The choice of the initial and final states should be a matter of physical consideration (see appendix 1). In this paper we will deal with the formal scheme suitable for a rather wide class of initial and final states.

We now write down some general assumptions, related to the construction of these states, which we use explicitly.

We suppose that the sets of creation and annihilation operators for the charged particles $\{\alpha_n^\dagger,\alpha_n\}$ and antiparticles $\{\beta_n^\dagger,\beta_n\}$ at the moment of time t_1 are given (and $[\alpha_n,\alpha_{n'}^\dagger]_+=[\beta_n,\beta_{n'}^\dagger]_+=\delta_{n,n'}$, $[\alpha_n,\alpha_{n'}]_+=[\beta_n,\beta_{n'}]_+=0$).

There is the vacuum vector $|0\rangle_{\rm in}$ for these operators ($\forall n$, $\alpha_n|0\rangle_{\rm in}=\beta_n|0\rangle_{\rm in}=0$) in the original Hilbert space, where the Hamiltonian $\mathscr{H}(t)$ is defined.

The set of operators $\{\alpha_n^\dagger,\alpha_n,\beta_n^\dagger,\beta_n\}$ is generated by the complete and orthonormal set of spinors $\{{}_\pm\phi_n(\boldsymbol{x})\}$, where a suffix + denotes a particle and − an antiparticle:

$$\psi(\boldsymbol{x})=\sum_n(\alpha_n\,{}_+\phi_n(\boldsymbol{x})+\beta_n^\dagger\,{}_-\phi_n(\boldsymbol{x})).\tag{4}$$

Here $\psi(\boldsymbol{x})$ is the spinor field operator in the Schrödinger picture.

Similarly the sets of operators of the charged particles $\{a_m^\dagger, a_m\}$ and antiparticles $\{b_m^\dagger, b_m\}([a_m, a_{m'}^\dagger]_+ = [b_m, b_{m'}^\dagger]_+ = \delta_{m,m'}, [a_m, a_{m'}]_+ = [b_m, b_{m'}]_+ = 0)$ at the moment of time t_2 are given, there is the vacuum vector $|0\rangle_{\text{out}}(\forall m, a_m|0\rangle_{\text{out}} = b_m|0\rangle_{\text{out}} = 0)$, operators $\{a_m^\dagger, a_m, b_m^\dagger, b_m\}$ are generated by the spinors $\{{}^{\pm}\phi_m(\boldsymbol{x})\}$ in accordance with the expansion:

$$\psi(\boldsymbol{x}) = \sum_m \{a_m\,{}^{+}\phi_m(\boldsymbol{x}) + b_m^\dagger\,{}^{-}\phi_m(\boldsymbol{x})\}. \tag{5}$$

Conditions for the orthonormality and completeness of the set of spinors $\{{}_{\pm}\phi_n(\boldsymbol{x})\}$ have the following form:

$$({}_{\pm}\phi_m, {}_{\pm}\phi_{n'}) = \delta_{n,n'}, \qquad ({}_{\pm}\phi_n, {}_{\mp}\phi_{n'}) = 0, \qquad (\phi, \psi) = \int \phi^\dagger(\boldsymbol{x})\psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x},$$
$$\sum_n ({}_{+}\phi_n(\boldsymbol{x})\,{}_{+}\phi_n^\dagger(\boldsymbol{x}') + {}_{-}\phi_n(\boldsymbol{x})_{-}\phi_n^\dagger(\boldsymbol{x})) = \delta(\boldsymbol{x} - \boldsymbol{x}'). \tag{6}$$

Similar conditions can be written for the set of spinors $\{{}^{\pm}\phi_m(\boldsymbol{x})\}$.

The requirement for the existence of the vacuum vectors $|0\rangle_{\text{in}}$ and $|0\rangle_{\text{out}}$ imposes certain restrictions on the choice of sets $\{{}_{\pm}\phi_n(\boldsymbol{x})\}$ and $\{{}^{\pm}\phi_m(\boldsymbol{x})\}$. These restrictions are discussed in appendix 1.

Thus the probability amplitude for an arbitrary process with initial and final states containing given numbers of charged particles, antiparticles and photons can be written in the form:

$$M_{\text{i}\to\text{f}} = {}_{\text{out}}\langle\tilde{0}|\tilde{a}_m \ldots \tilde{b}_s \ldots c_{\kappa\lambda} \ldots S(t_2, t_1)c_{\kappa\nu}^\dagger \ldots \beta_n^\dagger \ldots \alpha_l^\dagger \ldots |0\rangle_{\text{in}}, \tag{7}$$

where

$$\tilde{a}_m = U_0^{-1}(t_2, t_1)a_m U_0(t_2, t_1), \qquad \tilde{b}_m = U_0^{-1}(t_2, t_1)b_m U_0(t_2, t_1),$$
$$|\tilde{0}\rangle_{\text{out}} = U_0^{-1}(t_2, t_1)|0\rangle_{\text{out}}, \tag{8}$$

and $c_{\kappa\lambda}^\dagger$, $c_{\kappa\lambda}$ are the photon creation and annihilation operators.

In the matrix elements (7) the vacuum vectors and the creation and annihilation operators which are placed on the left and on the right of the S matrix are different from each other (see appendix 3). It is this that distinguishes the matrix elements (7) from the case when either no external field is involved or the latter does not produce pairs, and initial and final states are chosen in a special way.

That is why a result cannot be obtained using conventional computational techniques, based on the reduction of the S matrix to normal form relative to one vacuum. The main idea which allows us to obtain an analogue of conventional perturbation theory in quantum electrodynamics is to express any operators of the spinor field, and specifically the S matrix, only through creation $(\tilde{a}^\dagger, \tilde{b}^\dagger)$ and annihilation (α, β) operators, so that all the $\tilde{a}^\dagger$, $\tilde{b}^\dagger$ can be placed on the left of all the α, β. The correct computational techniques will be discussed later.

3. Reduction of operators to a generalised normal form

Define the generalised normal form of spinor field operators as a form where they are expressed only in terms of the creation $(\tilde{a}^\dagger, \tilde{b}^\dagger)$ and annihilation (α, β) operators, also the creation operators $\tilde{a}^\dagger$, $\tilde{b}^\dagger$ are placed on the left of the operators α, β.

The generalised normal product $\tilde{N}(\ldots)$ of the spinor field operators will be called the product of these operators reduced to the generalised normal form; the anticommutators are equal to zero in the process of reduction. The expressions

$$\underline{AB} = AB - \tilde{N}(AB), \qquad \overline{AB} = T(AB) - \tilde{N}(AB)$$

are called the generalised pairing and the generalised chronological pairing, respectively.

In accordance with the definition, to reduce operators to the generalised normal form it is necessary to express all the operators only in terms of $\tilde{a}^\dagger$, $\tilde{b}^\dagger$ and α, β. This can be done if the general connections between the operators $\{\alpha^\dagger, \alpha, \beta^\dagger, \beta\}$ and $\{\tilde{a}^\dagger, \tilde{a}, \tilde{b}^\dagger, \tilde{b}\}$ are established. To do so we will use the following. Since the operators $\psi(x)$, already discussed in connection with (2), satisfy the Dirac equation in the external field with the operator initial conditions $\psi(x)|_{t=t_1} = \psi(\boldsymbol{x})$ they may be given in the form:

$$\psi(x) = \int G(x, x_1)\psi(\boldsymbol{x}_1)\,\mathrm{d}\boldsymbol{x}_1, \tag{9}$$

where $G(x, x')$ is the propagator for the Dirac equation in the external field (Dirac 1937). It satisfies the Dirac equation and the condition $G(x, x')|_{x^0 = x^{0\prime}} = \delta(\boldsymbol{x} - \boldsymbol{x}')$. The function $G(x, x')$ may be constructed using any complete and orthonormal system of solutions $\{\phi_k(x)\}$ of the Dirac equation in the usual way:

$$G(x, x') = \sum_k \phi_k(x)\phi_k^\dagger(x').$$

The obvious relations $G^\dagger(x, x') = G^{-1}(x, x') = G(x', x)$ are valid for this function. It should be noted that $G(x, x')$ is the anticommutator of spinor field operators in the interaction picture (2).

From (2) and (9) we find the following relations:

$$U_0^{-1}(t, t')\psi(\boldsymbol{x})U_0(t, t') = \int G(x, x')\psi(\boldsymbol{x}')\,\mathrm{d}\boldsymbol{x}', \tag{10}$$

where t and t' are arbitrary moments of time. By substituting the expansion (4) in the right-hand side and the expansion (5) in the left-hand side of (10) and assuming $t' = t_1$, $t = t_2$ we get

$$\begin{aligned} \tilde{a} &= G({}^+|_+)\alpha + G({}^+|_-)\beta^\dagger, & \tilde{b}^\dagger &= G({}^-|_+)\alpha + G({}^-|_-)\beta^\dagger \\ \tilde{a}^\dagger &= \alpha^\dagger G({}_+|^+) + \beta G({}_-|^+), & \tilde{b} &= \alpha^\dagger G({}_+|^-) + \beta G({}_-|^-), \end{aligned} \tag{11}$$

where

$$G({}^\pm|_\pm)_{mn} = \int {}^\pm\phi_m^\dagger(\boldsymbol{x}_2)G(x_2, x_1)_\pm\phi_n(\boldsymbol{x}_1)\,\mathrm{d}\boldsymbol{x}_1\,\mathrm{d}\boldsymbol{x}_2,$$

$$G({}_\pm|^\pm)_{nm} = \int {}_\pm\phi_n^\dagger(\boldsymbol{x}_1)G(x_1, x_2)^\pm\phi_m(\boldsymbol{x}_2)\,\mathrm{d}\boldsymbol{x}_1\,\mathrm{d}\boldsymbol{x}_2, \qquad G({}^\pm|_\pm)^\dagger = G({}_\pm|^\pm).$$

By substituting the expansion (5) in the right-hand side and the expansion (4) in the left-hand side of (10) and assuming $t = t_1$, $t' = t_2$ we obtain the relations inverse to (11):

$$\begin{aligned} \alpha &= G({}_+|^+)\tilde{a} + G({}_+|^-)\tilde{b}^\dagger, & \beta^\dagger &= G({}_-|^+)\tilde{a} + G({}_-|^-)\tilde{b}^\dagger, \\ \alpha^\dagger &= \tilde{a}^\dagger G({}^+|_+) + \tilde{b}G({}^-|_+), & \beta &= \tilde{a}^\dagger G({}^+|_-) + \tilde{b}G({}^-|_-). \end{aligned} \tag{12}$$

We now introduce the notation for the relative probability amplitudes of the processes which are of zeroth order with respect to the electron–photon interaction as follows:

$$w(\overset{+}{m}\ldots\bar{s}\ldots|\bar{n}\ldots\overset{+}{l}\ldots)={}_{\text{out}}\langle\tilde{0}|\tilde{a}_m\ldots\tilde{b}_s\ldots\beta_n^\dagger\ldots\alpha_l^\dagger\ldots|0\rangle_{\text{in}}C_{\text{v}}^{-1},\tag{13}$$

where

$$C_{\text{v}}={}_{\text{out}}\langle 0|U_0(t_2,t_1)|0\rangle_{\text{in}}={}_{\text{out}}\langle\tilde{0}|0\rangle_{\text{in}}\tag{14}$$

is the probability amplitude for the vacuum to remain the vacuum to zeroth order with respect to the electron–photon interaction. The simplest amplitudes, which correspond to single-particle scattering, annihilation and creation of pairs, are readily calculated by direct use of relations (11) and (12) (Bagrov *et al* 1975):

$$\begin{aligned}
&w(\overset{+}{m}|\overset{+}{n})=G^{-1}({}_+|^+)_{mn},\qquad w(\bar{m}|\bar{n})=G^{-1}({}^-|_-)_{nm},\\
&w(0|\bar{n}\overset{+}{l})=\{G({}_-|^+)G^{-1}({}_+|^+)\}_{nl}=-\{G^{-1}({}^-|_-)G({}^-|_+)\}_{nl},\\
&w(\overset{+}{m}\bar{s}|0)=\{G^{-1}({}_+|^+)G({}_+|^-)\}_{ms}=-\{G({}^+|_-)G^{-1}({}^-|_-)\}_{ms}.
\end{aligned}\tag{15}$$

From (11), (12) and (15) we get the following relations:

$$\begin{aligned}
\tilde{a}_m&=\sum_n w(\overset{+}{m}|\overset{+}{n})\alpha_n-\sum_s w(\overset{+}{m}\bar{s}|0)\tilde{b}_s^\dagger,\\
\tilde{b}_m&=\sum_n w(\bar{m}|\bar{n})\beta_n+\sum_s w(\overset{+}{s}\bar{m}|0)\tilde{a}_s^\dagger,\\
\alpha_n^\dagger&=\sum_m w(\overset{+}{m}|\overset{+}{n})\tilde{a}_m^\dagger-\sum_l w(0|\bar{l}\overset{+}{n})\beta_l,\\
\beta_n^\dagger&=\sum_m w(\bar{m}|\bar{n})\tilde{b}_m^\dagger+\sum_l w(0|\bar{n}\overset{+}{l})\alpha_l.
\end{aligned}\tag{16}$$

Relations (16) allow us to express unambiguously all the spinor field operators as functions of the creation $(\tilde{a}^\dagger, \tilde{b}^\dagger)$ and annihilation (α, β) operators only. We will, for example, do this for the operators $\psi(x)$ and $\bar{\psi}(x)$. To do so we substitute the expansion (4) in the right-hand side of (9) to find

$$\psi(x)=\sum_n(\alpha_n\,{}_+\phi_n(x)+\beta_n^\dagger\,{}_-\phi_n(x)),\tag{17}$$

where ${}_\pm\phi_n(x)=\int G(x,x_1)\,{}_\pm\phi_n(\boldsymbol{x}_1)\,\mathrm{d}\boldsymbol{x}_1$ are the solutions of the Dirac equation in the field with the initial conditions given at $t=t_1$. By using definition (2) and the obvious connection $\psi(x)=\int G(x,x_2)\psi(x_2)\,\mathrm{d}\boldsymbol{x}_2$ and the expansion (5), we find:

$$\psi(x)=\sum_m(\tilde{a}_m\,{}^+\phi_m(x)+\tilde{b}^\dagger\,{}^-\phi_m(x)),\tag{18}$$

where ${}^\pm\phi_m(x)=\int G(x,x_2)\,{}^\pm\phi_m(\boldsymbol{x}_2)\,\mathrm{d}\boldsymbol{x}_2$ are the solutions of the Dirac equation in the field with the initial conditions given at $t=t_2$.

By substituting $\alpha^\dagger$, $\beta^\dagger$, $\tilde{a}$, $\tilde{b}$ from (16) into (17), (18) and into the Dirac conjugate expressions we get the necessary form for $\psi(x)$ and $\bar{\psi}(x)$:

$$\psi(x)=\psi^{(-)}(x)+\psi^{(+)}(x), \qquad \bar{\psi}(x)=\bar{\psi}^{(-)}(x)+\bar{\psi}^{(+)}(x),$$

$$\psi^{(-)}(x)=\sum_n {}_+\psi_n(x)\alpha_n, \qquad \bar{\psi}^{(-)}(x)=\sum_n {}_-\bar{\psi}_n(x)\beta_n,$$

$$\psi^{(+)}(x)=\sum_m {}^-\psi_m(x)\tilde{b}^\dagger_m, \qquad \bar{\psi}^{(+)}(x)=\sum_m {}^+\bar{\psi}(x)\tilde{a}^\dagger_m,$$

$${}_+\psi_n(x)={}_+\phi_n(x)+\sum_l w(0|\bar{l}\overset{+}{n})_-\phi_l(x)=\sum_m w(\overset{+}{m}|\overset{+}{n})^+\phi_m(x),$$

$${}^-\psi_m(x)={}^-\phi_m(x)-\sum_s w(\overset{+}{s}\bar{m}|0)^+\phi_s(x)=\sum_n w(\bar{m}|\bar{n})_-\phi_n(x),$$

$${}_-\bar{\psi}_n(x)={}_-\bar{\phi}_n(x)-\sum_l w(0|\bar{n}\overset{+}{l})_+\bar{\phi}_l(x)=\sum_m w(\bar{m}|\bar{n})^-\bar{\phi}_m(x),$$

$${}^+\bar{\psi}_m(x)={}^+\bar{\phi}_m(x)+\sum_s w(\overset{+}{m}\bar{s}|0)^-\bar{\phi}_s(x)=\sum_n w(\overset{+}{m}|\overset{+}{n})_+\bar{\phi}_n(x).$$

Once the spinor field operators have been expressed in the form of functions which depend only on $\tilde{a}^\dagger$, $\tilde{b}^\dagger$, α, β, they can be reduced to generalised normal form by using known versions of the commonly used Wick's theorem. For this purpose it is necessary to calculate the following anticommutators and generalised pairings:

$$[\alpha_n, \tilde{b}^\dagger_m]_+=[\beta_n, \tilde{a}^\dagger_m]_+=0,$$

$$[\alpha_n, \tilde{a}^\dagger_m]_+=G({}_+|{}^+)_{nm}, \qquad [\beta_n, \tilde{b}^\dagger_m]_+=G({}^-|{}_-)_{mn},$$

$$[\psi^{(-)}, \bar{\psi}^{(-)}]_+=[\psi^{(+)}, \bar{\psi}^{(+)}]_+=[\psi^{(-)}, \psi^{(+)}]_+=[\bar{\psi}^{(-)}, \bar{\psi}^{(+)}]_+=0,$$

$$[\psi^{(-)}(x), \bar{\psi}^{(+)}(x')]_+=\sum_{n,m} {}^+\phi_m(x)w(\overset{+}{m}|\overset{+}{n})_+\bar{\phi}_n(x')=\frac{1}{i}\tilde{S}^-(x,x'),$$

$$[\psi^{(+)}(x), \bar{\psi}^{(-)}(x')]_+=\sum_{n,m} {}_-\phi_n(x)w(\bar{m}|\bar{n})^-\bar{\phi}_m(x')=\frac{1}{i}\tilde{S}^+(x,x'), \tag{19}$$

$$\underbrace{\tilde{a}_m\tilde{b}_s}={}_{\text{out}}\langle\tilde{0}|\tilde{a}_m\tilde{b}_s|0\rangle_{\text{in}}C_v^{-1}=w(\overset{+}{m}\bar{s}|0),$$

$$\underbrace{\beta^\dagger_n\alpha^\dagger_l}={}_{\text{out}}\langle\tilde{0}|\beta^\dagger_n\alpha^\dagger_l|0\rangle_{\text{in}}C_v^{-1}=w(0|\bar{n}\overset{+}{l}),$$

$$\underbrace{\tilde{b}_m\beta^\dagger_n}={}_{\text{out}}\langle\tilde{0}|\tilde{b}_m\beta^\dagger_n|0\rangle_{\text{in}}C_v^{-1}=w(\bar{m}|\bar{n}),$$

$$\underbrace{\tilde{a}_m\alpha^\dagger_n}={}_{\text{out}}\langle\tilde{0}|\tilde{a}_m\alpha^\dagger_n|0\rangle_{\text{in}}C_v^{-1}=w(\overset{+}{m}|\overset{+}{n}),$$

$$\underbrace{\tilde{a}_m\beta^\dagger_n}=\underbrace{\tilde{b}_m\alpha^\dagger_n}=0$$

$$\overbrace{\psi(x)\bar{\psi}(x')}={}_{\text{out}}\langle\tilde{0}|T\psi(x)\bar{\psi}(x')|0\rangle_{\text{in}}C_v^{-1}=\frac{1}{i}\tilde{S}^c(x,x'), \tag{20}$$

$$\tilde{S}^c(x,x')=\begin{cases}\tilde{S}^-(x,x'), & x^0>x^{0\prime},\\ -\tilde{S}^+(x,x'), & x^{0\prime}>x^0.\end{cases}$$

With the help of these expressions it is not difficult to establish that

$$_{+}\psi_n(x)=\frac{1}{\mathrm{i}}\int \tilde{S}^-(x,x_1)\gamma^0\,_{+}\phi_n(\boldsymbol{x}_1)\,\mathrm{d}\boldsymbol{x}_1, \tag{21}$$

$$^{-}\psi_m(x)=\frac{1}{\mathrm{i}}\int \tilde{S}^+(x,x_2)\gamma^{0\,-}\phi_m(\boldsymbol{x}_2)\,\mathrm{d}\boldsymbol{x}_2, \tag{22}$$

$$_{-}\bar{\psi}(x)=\frac{1}{\mathrm{i}}\int {}_{-}\phi_n^{\dagger}(\boldsymbol{x}_1)\tilde{S}^+(x_1,x)\,\mathrm{d}\boldsymbol{x}_1, \tag{23}$$

$$^{+}\bar{\psi}_m(x)=\frac{1}{\mathrm{i}}\int {}^{+}\phi_m^{\dagger}(\boldsymbol{x}_2)\tilde{S}^-(x_2,x)\,\mathrm{d}\boldsymbol{x}_2. \tag{24}$$

Evidently $\tilde{S}^c(x,x')$ is the generalisation of the Feynman causal Green function for the case of an arbitrary external field, creating pairs. One can check that $\tilde{S}^c(x,x')$ satisfies the Dirac equation for the Green function in the external field:

$$(\hat{\mathscr{P}}-m)S(x,x')=-\delta(x-x'), \qquad \mathscr{P}_\mu=\mathrm{i}\partial_\mu-e\tilde{\mathscr{A}}_\mu(x). \tag{25}$$

Compare the function $\tilde{S}^c(x,x')$ with the Feynman propagation function in an external electromagnetic field $S^c(x,x')$. Note that what we usually understand by the Feynman propagation function is the function which satisfies equation (25) and which is represented in the form of the formal series (Feynman 1949, Bogolubov and Shirkov 1973):

$$S^c(x,x')=S_0^c(x,x')-e\int S_0^c(x,x_1)\hat{\mathscr{A}}(x_1)S_0^c(x_1,x')\,\mathrm{d}x_1$$
$$+e^2\int S_0^c(x,x_1)\hat{\mathscr{A}}(x_1)S_0^c(x_1,x_2)\hat{\mathscr{A}}(x_2)S_0^c(x_2,x')\,\mathrm{d}x_1\,\mathrm{d}x_2+\ldots \tag{26}$$

where $S_0^c(x,x')$ is the Feynman causal Green function of the free spinor field:

$$S_0^c(x,x')=\mathrm{i}\langle 0|T\psi_0(x)\bar{\psi}(x')|0\rangle=-\frac{1}{(2\pi)^4}\int\frac{(m+\hat{p})\exp[-\mathrm{i}p(x-x')]}{p^2-m^2+\mathrm{i}\epsilon}\,\mathrm{d}^4p.$$

$\psi_0(x)$, $\bar{\psi}_0(x)$ are the operators of the usual interaction picture, and $|0\rangle$ is the vacuum of the free particles.

We obtain the formal expansion of the function $\tilde{S}^c(x,x')$ in a series in powers of the external field if we write:

$$\psi(x)=S_0^{-1}(t,t_1)\psi_0(x)S_0(t,t_1)$$

where

$$S_0(t,t_1)=\exp[\mathrm{i}\mathscr{H}_e(t-t_1)]U_0(t,t_1)=T\exp\left(-\mathrm{i}\int_{t_1}^{t} j_0^\mu(x)\tilde{\mathscr{A}}_\mu(x)\,\mathrm{d}x\right),$$

$\mathscr{H}_e$ is the free electron–positron field Hamiltonian, $j_0^\mu(x)$ is the current operator in the interaction picture. Then:

$$\tilde{S}^c(x,x')=\mathrm{i}\,_{\mathrm{out}}\langle\tilde{\tilde{0}}|T\psi_0(x)\bar{\psi}_0(x')S_0(t_2,t_1)|0\rangle_{\mathrm{in}}C_v^{-1},$$
$$|\tilde{\tilde{0}}\rangle_{\mathrm{out}}=\exp[\mathrm{i}\mathscr{H}_e(t_2-t_1)]|0\rangle_{\mathrm{out}}, \qquad C_v={}_{\mathrm{out}}\langle\tilde{\tilde{0}}|S_0(t_2,t_1)|0\rangle_{\mathrm{in}}.$$

By using the techniques considered above to reduce the generalised normal form relative to the two different vacuums $|0\rangle_{\mathrm{in}}$ and $|\tilde{\tilde{0}}\rangle_{\mathrm{out}}$, one may obtain the series (26) for $\tilde{S}^{\mathrm{c}}(x, x')$ with $S^{\mathrm{c}}_0(x, x')$ replaced by $\tilde{S}^{\mathrm{c}}_0(x, x')$:

$$\tilde{S}^{\mathrm{c}}_0(x, x') = {}_{\mathrm{out}}\langle\tilde{\tilde{0}}|T\psi_0(x)\bar{\psi}_0(x')|0\rangle_{\mathrm{in}}.$$

Therefore these formal considerations show that the functions S^{c} and $\tilde{S}^{\mathrm{c}}$ are different for a general case. They coincide deliberately if the vacuum vectors of the initial and final states coincide and are those of free particles. Note that in the Schwinger's (1954) well known work the Green function in an external field is considered. Its explicit form given in Schwinger (1954) is as follows:

$$G(x, x') = \mathrm{i}\langle 0|T\psi(x)\bar{\psi}(x')|0\rangle. \tag{27}$$

However, Schwinger noted clearly that he was discussing only fields creating no pairs. In this case expressions (20) and (27) do coincide. Detailed consideration of the proper time method as applied for obtaining the Green functions in an external field (Schwinger 1954) shows that the method contains some ambiguity which disappears as far as the external field creating no pairs is concerned. When dealing with the field creating pairs it is necessary to use either additional boundary conditions or the explicit form of $\tilde{S}^{\mathrm{c}}$, given by expression (20). However, considerable simplifications are possible in specific cases. It is obvious, for instance, that the Green function in a constant and uniform electric field can be found from the corresponding Green function in a constant and uniform magnetic field by means of the substitution $H \to \mathrm{i}E$. The expression to be obtained can be brought into the form (20) (Nikishov 1969).

4. Correspondence rules

It is obviously convenient to represent the S matrix in the generalised normal form for the calculations of matrix elements (7) from the S matrix (the generalised normal form of the electromagnetic field operators coincides with their usual normal form). This can be done with the help of Wick's theorem for the T products, where the normal products and chronological pairings are replaced by their generalised analogues. It is useful first to represent the current operator in the generalised normal form:

$$\begin{aligned} j^{\mu}(x) &= e\tilde{N}\bar{\psi}(x)\gamma^{\mu}\psi(x) + \mathcal{J}^{\mu}(x), \\ \mathcal{J}^{\mu}(x) &= {}_{\mathrm{out}}\langle\tilde{0}|j^{\mu}(x)|0\rangle_{\mathrm{in}} = \tfrac{1}{2}\mathrm{i}e\,\mathrm{Tr}\,\gamma^{\mu}(\tilde{S}^{\mathrm{c}}(x+0, x) + \tilde{S}^{\mathrm{c}}(x, x+0)). \end{aligned} \tag{28}$$

Thus the problem is reduced to the calculation of the matrix elements of the generalised normal products:

$${}_{\mathrm{out}}\langle\tilde{0}|\tilde{a}_m \ldots \tilde{b}_s \ldots c_{\kappa\lambda} \ldots \tilde{N}(\ldots)c^{\dagger}_{\varkappa\nu} \ldots \beta^{\dagger}_n \ldots \alpha^{\dagger}_l \ldots|0\rangle_{\mathrm{in}}.$$

Evidently this matrix element is non-zero if the sum of the number of particles for each field at the initial and final states is greater than or equal to the number of operator functions of the given field in the generalised normal product.

Consider the case when for each field operator $\psi(x), \bar{\psi}(x)$, $\mathcal{A}_{\mu}(x)$ from the generalised normal product one can find the corresponding operator $\alpha^{\dagger}$, $\beta^{\dagger}$, $c^{\dagger}$ from the initial state, or $\tilde{a}$, $\tilde{b}$, c from the final state, which will cancel it as the result of commutation. Such a matrix element will be presented by means of Feynman diagrams with the following, partly modified, correspondence rules.

(i) The factor ${}_{+}\psi_n(x)({}^{+}\bar{\psi}_m(x))$ in the matrix element, defined by the expressions (21) ((24)), corresponds to the electron with the quantum number n (m) at the initial (final) state.

(ii) The factor ${}_{-}\bar{\psi}_n(x)({}^{-}\psi_m(x))$ in the matrix element, defined by the expressions (23) ((22)), corresponds to the positron with the quantum number n (m) at the initial (final) state.

(iii) The pairing $\mathrm{i}^{-1}\tilde{S}^c(x, x')$, defined by expression (20), corresponds to the internal electron lines, directed from the point x' to the point x.

(iv) The c-number current $\mathcal{J}(x)$, defined by expression (28), corresponds to the closed electron lines.

(v) The contribution from any diagram contains as a factor the probability amplitude of the vacuum remaining the vacuum C_v.

The rest of the correspondence rules remain unchanged for this case (Bogolubov and Shirkov 1973).

Now consider the case when the number of the spinor operators at the initial and final states is greater than that which is necessary for the compensation of the generalised normal product. This matrix element is equal to the products of the contributions coming from the Feynman diagrams and the factors $w(\overset{+}{m} \ldots \bar{s} \ldots |\overset{+}{n} \ldots \bar{l} \ldots)$. The Feynman contributions arise due to the 'interaction' of the generalised normal product with the operators of the initial and final states. The w come from the non-compensated operators of the creation and annihilation of these states. The contributions of the Feynman diagrams are calculated with the use of the modifications listed above. The method of computation of the amplitudes $w(\overset{+}{m} \ldots \bar{s} \ldots |\overset{+}{n} \ldots \bar{l} \ldots)$ is discussed in § 5.

5. Relative probability amplitudes for zeroth-order processes

Relative probability amplitudes for processes which are of zeroth order in the electron–photon interaction (13) are readily calculated if the product $\tilde{a}_m \ldots \tilde{b}_s \ldots \beta_n^\dagger \ldots \alpha_l^\dagger \ldots$ of the operators is reduced to the generalised normal form with the help of Wick's theorem and the pairings (19). Evidently, since ${}_{\text{out}}\langle\tilde{0}|\tilde{N}(\ldots)|0\rangle_{\text{in}} = 0$, the matrix element (13) is equal to the sum of all the possible pairings of the operators $\tilde{a}_m \ldots \tilde{b}_s \ldots \beta_n^\dagger \ldots \alpha_l^\dagger \ldots$ without the $\tilde{N}$-products, i.e. it is expressed according to (19) only in terms of the sum of the products of the amplitudes for the processes of scattering, annihilation and creation of pairs taken with the corresponding signs, determined according to Wick's theorem. For example, the probability amplitude for electron scattering accompanied by the creation of a pair is expressed as follows:

$$w(\overset{+}{m}\overset{+}{s}\bar{k}|\overset{+}{n}) = w(\overset{+}{s}\bar{k}|0)w(\overset{+}{m}|\overset{+}{n}) - w(\overset{+}{m}\bar{k}|0)w(\overset{+}{s}|\overset{+}{n}).$$

As we have already seen in § 4 the contribution of any diagram contains as a factor the probability amplitude for the vacuum to remain the vacuum (14). We will now discuss reasons enabling us to determine this quantity using the initial constructions, i.e. the propagator $G(x, x')$ and the spinors of the initial and final states.

In the assumptions, discussed in appendix 3, a unitary operator V exists such that

$$\tilde{a}_m = V^{-1}\alpha_m V, \qquad \tilde{b}_m = V^{-1}\beta_m V, \qquad |0\rangle_{\text{out}} = V^{-1}|0\rangle_{\text{in}}. \tag{29}$$

Evidently from (14) and (19) it follows that C_v is the expectation value of the operator V calculated over the vacuum $|0\rangle_{in}$:

$$C_v = {}_{in}\langle 0|V|0\rangle_{in}. \tag{30}$$

The operator V can be determined from the relations (11) by substituting $\{\tilde{a}^\dagger, \tilde{a}, \tilde{b}^\dagger, \tilde{b}\}$ in them according to (29). Using the known formula

$$\exp(\tau\mathcal{K})\mathcal{L}\exp(-\tau\mathcal{K}) = \sum_{n=0}^{\infty}\frac{\tau^n}{n!}[\mathcal{K}, [\mathcal{K}, \ldots [\mathcal{K}, \mathcal{L}]_- \ldots]_-, \tag{31}$$

it is easy to show that it is always possible to satisfy these relations by choosing V in the following form:

$$V = \exp(\alpha^\dagger A\alpha + \alpha^\dagger B\beta^\dagger + \beta C\alpha + \beta D\beta^\dagger). \tag{32}$$

By such a choice of V one can ensure its unitarity in an uncontradictory way. Further, one can verify that the set of commutators of the four quadratic forms $\alpha^\dagger A\alpha$, $\alpha^\dagger B\beta^\dagger$, $\beta C\alpha$, $\beta D\beta^\dagger$ generates the set of quadratic forms of the same type, i.e. with the analogous arrangement of creation and annihilation operators. In this case it is always possible to find matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ so that the expression (32) can be rewritten in the form:

$$V = \exp(\alpha^\dagger\tilde{B}\beta^\dagger)\exp(\alpha^\dagger\tilde{A}\alpha)\exp(\beta\tilde{D}\beta^\dagger)\exp(\beta\tilde{C}\alpha). \tag{33}$$

One may verify this by using the known expansions of the operator exponentials (Kirzhnitz 1963). The explicit forms of the matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ can be found by substituting (33) in (29), (11) and using the formula (31). For instance, the required matrix $\tilde{D}$ is connected with the matrix $G({}^-|{}_-)$ by the relation $G({}^-|{}_-) = \exp(\tilde{D})$.

Substituting (33) in (30) one gets

$$C_v = \exp(\mathrm{Tr}\ln G({}^-|{}_-)) = \det G({}^-|{}_-). \tag{34}$$

The derivation of formula (34) which we have presented is not the only one possible. For example, S P Gavrilov noted that it is possible to arrive at the same conclusion by using a combinatoric method.

6. Conclusions

The results of this work allow us to use known methods of perturbation theory and diagrammatic techniques with little modification to consider processes in an arbitrary electromagnetic field creating pairs. The modifications are reduced merely to some new definitions of all the electron lines, both internal and external. It should be noted that the causal Green functions, corresponding to the internal lines, have different forms for problems with different sets of initial and final states. Actually they 'feel' the vacuum states of the initial and final particles.

Note that the method suggested can be useful for the consideration of quantum processes in external gravitational fields where, in the general case, the vacuum vectors of the initial and final states are different.

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Appendix 1

The problem of the choice of initial and final states in theories with external electromagnetic or gravitational fields has no general solution yet. This problem is obviously associated with the definition of the vacuum in an external field. We think that there is the following possible approach to the solution of the problem.

The vector minimising the mean value of the Hamiltonian $\mathscr{H}(t)$ at the given moment of time t will be called the vacuum vector $|0\rangle_t$ at the moment of time t;

$$ {}_t\langle 0|\mathscr{H}(t)|0\rangle_t = \text{minimum value}. \tag{A.1}$$

In connection with that stated above we consider the eigenvalue problem

$$\mathscr{H}(t)|\chi_n\rangle_t = E_n(t)|\chi_n\rangle_t \tag{A.2}$$

where t is a parameter. If the spectrum $E_n(t)$ is bounded from below and $E_0(t)$ is the smallest value, then it is not difficult to see that $|\chi_0\rangle_t$ is the solution of the functional equation (A.2) and, therefore, it could be called the vacuum vector at the moment of time t. Obviously $E_n(t)$ gives the possible values of energy of the system at the moment of time t, and $|\chi_n\rangle_t$ is the state with the given energy at the moment of time t. The vacuum remains if $|\chi_0\rangle_t$ at any moment of time coincides to within the phase factor with $U(t, t_1)|\chi_0\rangle_{t_1}$, where $U(t, t')$ is the evolution operator of the total system.

Practically, the exact solution of the problem (A.2) is not possible. In a strong external field it is reasonable to consider the approximate problem with the Hamiltonian $\mathscr{H}_0(t)$ (see (1)). The solution of the latter problem is possible if the eigenvalue problem for the Dirac Hamiltonian in an external field is solved:

$$\begin{aligned} &\mathscr{H}_D(t)\,{}_\pm\phi_n(\boldsymbol{x}, t) = \mathscr{E}_n^{(\pm)}(t)\,{}_\pm\phi_n(\boldsymbol{x}, t),\\ &\mathscr{H}_D(t) = -\mathrm{i}\boldsymbol{\alpha}\cdot\nabla - e\boldsymbol{\alpha}\cdot\tilde{\boldsymbol{\mathscr{A}}}(x) + e\tilde{\mathscr{A}}_0(x) + m\beta. \end{aligned} \tag{A.3}$$

In addition, the following conditions must be valid:

(*a*) $\mathscr{E}_n^{(+)} > 0$, $\mathscr{E}_n^{(-)} < 0$ and there is a gap between positive and negative levels.

(*b*) Spinors $\{{}_\pm\phi_n(\boldsymbol{x}, t)\}$ at the given moment of time t satisfy the orthogonality and completeness conditions (6).

(*c*) The condition

$$\sum_{n,m} (|({}_+\phi_n, {}_-\psi_m^0)|^2 + |({}_-\phi_n, {}_+\psi_m^0)|^2) < \infty \tag{A.4}$$

is valid, where $\{{}_\pm\psi_m^0(\boldsymbol{x})\}$ are the solutions of equation (A.3) in the absence of the external field.

Indeed, by using condition (*b*) and the expansion of the type (4), one can introduce the operators of creation and annihilation of particles $\{\alpha_n^\dagger(t), \alpha_n(t)\}$ and antiparticles

$\{\beta_n^\dagger(t), \beta_n(t)\}$. By using condition ($a$) it is not difficult to obtain the following form of the operator $\mathscr{H}_0(t)$:

$$\mathscr{H}_0(t) = \sum_n (\mathscr{E}_n^{(+)}\alpha_n^\dagger(t)\alpha_n(t) + |\mathscr{E}_n^{(-)}|\beta_n^\dagger(t)\beta_n(t)) + C(t),$$

where $C(t)$ is a constant, not an operator. Thus the vacuum vector $|0\rangle_t$ is the solution of the equation

$$\alpha_n(t)|0\rangle_t = \beta_n(t)|0\rangle_t = 0 \qquad \forall n. \tag{A.5}$$

This equation has a solution in the original Hilbert space, if the operators $\{\alpha_n^\dagger(t), \alpha_n(t), \beta_n^\dagger(t), \beta_n(t)\}$ are unitary equivalent to any operators of creation and annihilation for which there is a vacuum vector in the space concerned (Fridrichs 1953).

Let us choose, for instance, these operators to be the free particle operators $\{a_m^{0\dagger}, a_m^0, b_m^{0\dagger}, b_m^0\}$, generated by the set of spinors $\{_\pm\psi_m^0(\boldsymbol{x})\}$. By comparing the expansion of the operator $\psi(\boldsymbol{x})$ in spinors $\{_\pm\phi_n(\boldsymbol{x}, t)\}$ with the expansion in spinors $\{_\pm\psi_m^0(\boldsymbol{x})\}$ we find:

$$B_{n\lambda} = \sum_{\mu,\gamma} (\Phi_{n\lambda,\mu\gamma}A_{\mu\gamma} + \Psi_{n\lambda,\mu\gamma}A_{\mu\gamma}^\dagger), \qquad \lambda, \gamma = \pm 1 \tag{A.6}$$

where

$$B_{n+} = \alpha_n, \qquad B_{n-} = \beta_n, \qquad A_{\mu+} = a_\mu^0, \qquad A_{\mu-} = b_\mu^0,$$

$$\Phi_{n+,\mu+} = ({}_+\phi_n, {}_+\psi_\mu^0), \qquad \Phi_{n+,\mu-} = \Phi_{n-,\mu+} = 0, \qquad \Phi_{n-,\mu-} = ({}_-\phi_n, {}_-\psi_\mu^0)^*,$$

$$\Psi_{n+,\mu+} = \Psi_{n-,\mu-} = 0, \qquad \Psi_{n+,\mu-} = ({}_+\phi_n, {}_-\psi_\mu^0), \qquad \Psi_{n-,\mu+} = ({}_-\phi_n, {}_+\psi_\mu^0)^*.$$

It is evident that the canonical transformation (A.6) is a linear transformation. A theorem exists (Berezin 1965, Kiperman 1970), which holds for the linear canonical transformation of Fermi creation and annihilation operators. In accordance with this theorem, (A.6) is a proper transformation (i.e. $\alpha^\dagger(t)$, $\alpha(t)$, $\beta^\dagger(t)$, $\beta(t)$ and $a^{0\dagger}$, a^0, $b^{0\dagger}$, b^0 are unitary equivalent) if Ψ is the Hilbert–Schmidt operator. With our notation we arrive at condition (c).

The excited states $|\chi_n\rangle_t$ can be constructed in the usual way with respect to the vacuum $|0\rangle_t$ and to the corresponding creation and annihilation operators.

Appendix 2

Consider the problem of unitarity of the electron–positron field evolution operator $U_0(t_2, t_1)$ in an external electromagnetic field. This problem is uniquely connected with the problem of showing that the canonical transformation (11) of the operators $\{\alpha^\dagger, \alpha, \beta^\dagger, \beta\}$ to the operators $\{\tilde{a}^\dagger, \tilde{a}, \tilde{b}^\dagger, \tilde{b}\}$ is the proper transformation. Indeed, conditions (A.4), assumed for the spinors of the initial and final states, ensure the unitary equivalency of the operators $\{\alpha^\dagger, \alpha, \beta^\dagger, \beta\}$ and $\{a^\dagger, a, b^\dagger, b\}$. Therefore, from definitions (8) it follows that if $U_0(t_2, t_1)$ is the unitary operator, then there is an operator V, such that

$$\tilde{a}_m = V^{-1}\alpha_m V, \qquad \tilde{b}_m = V^{-1}\beta_m V, \qquad |\tilde{0}\rangle_{\text{out}} = V^{-1}|0\rangle_{\text{in}}, \tag{A.7}$$

and consequently the transformation (11) is the proper transformation. The reverse is also evident.

Let us investigate whether (11) is the proper transformation according to the theorems suggested by Berezin (1965) and Kiperman (1970), as it is done in appendix 1. Taking into account properties of matrices $G(^+|_-)$ and $G(^-|_+)$, defined earlier, we obtain the corresponding criterion

$$\mathrm{Tr}\{G(^+|_-)G(_-|^+)+G(^-|_+)G(_+|^-)\}<\infty. \tag{A.8}$$

We will show that the left-hand side of the inequality represents the total number of particles created by the field during the period of time (t_2-t_1). To do this we calculate probabilities of electron creation at the given quantum state $w(\overset{+}{m})$, and positron creation at the given quantum state $w(\bar{s})$, using the formulae (11) and assuming that $U_0(t_2, t_1)$ is the unitary operator:

$$w(\overset{+}{m})=\sum_{l,k=0}^{\infty}\sum_{\{m_l\}\{s_l\}}|_{\text{out}}\langle 0|a_m a_{m_1}\ldots a_{m_l}b_{s_1}\ldots b_{s_k}U_0(t_2,t_1)|0\rangle_{\text{in}}|^2$$
$$={}_{\text{in}}\langle 0|U_0^{-1}(t_2,t_1)a_m^\dagger a_m U_0(t_2,t_1)|0\rangle_{\text{in}}=\{G(^+|_-)G(_-|^+)\}_{mm}, \tag{A.9}$$

$$w(\bar{s})=\sum_{l,k=0}^{\infty}\sum_{\{m_l\}\{s_l\}}|_{\text{out}}\langle 0|a_{m_1}\ldots a_{m_l}b_s b_{s_1}\ldots b_{s_k}U_0(t_2,t_1)|0\rangle_{\text{in}}|^2$$
$$={}_{\text{in}}\langle 0|U_0^{-1}(t_2,t_1)b_s^\dagger b_s U_0(t_2,t_1)|0\rangle_{\text{in}}=\{G(^-|_+)G(_+|^-)\}_{ss}. \tag{A.10}$$

According to the Pauli principle expressions (A.9) and (A.10) are also the mean numbers of electrons and positrons created at the given quantum state. Thus the total numbers of electrons $w(+)$ and positrons $w(-)$ created during the period of time (t_2-t_1) are equal to, respectively:

$$w(+)=\mathrm{Tr}\,G(^+|_-)G(_-|^+), \qquad w(-)=\mathrm{Tr}\,G(^-|_+)G(_+|^-),$$

and the left-hand side of (A.8) really represents the total number of particles created. (It is possible to verify that $w(+)=w(-)$, so the law of conservation of charge is valid for this case.)

Thus we have proved that if $U_0(t_2, t_1)$ is the unitary operator, then the total number of particles created is not equal to infinity.

On the other hand, it is evident that this number is always the same for a system with a finite volume $\mathcal{V}$ during the finite time interval (t_2-t_1). If the external electromagnetic field is such that at $\mathcal{V}\to\infty$, or at $(t_2-t_1)\to\infty$, it creates an infinite number of pairs (for instance, the constant electric field), then according to the above discussion the evolution operator $U_0(t_2, t_1)$ is not the unitary operator.

It should be noted that the unitarity of the operator $U_0(t_2, t_1)$ has been proved by Schwinger (1954) and Nikishov (1974). However, the problem of the conservation of unitarity at $\mathcal{V}\to\infty$, $t_2-t_1\to\infty$, has not been investigated.

Appendix 3

Consider expressions (17) and (18) for the operator $\psi(x)$ at the moment $t=t_2$:

$$\psi(\boldsymbol{x}, t_2)=\sum_n (\alpha_n\, {}_+\phi_n(\boldsymbol{x}, t_2)+\beta_n^\dagger\, {}_-\phi_n(\boldsymbol{x}, t_2)),$$

$$\psi(\boldsymbol{x}, t_2)=\sum_m (\tilde{a}_m\, {}^+\phi_m(\boldsymbol{x})+\tilde{b}_m^\dagger\, {}^-\phi_m(\boldsymbol{x})).$$

Evidently, if we choose ${}^{\pm}\phi_m(\boldsymbol{x}) = {}_{\pm}\phi_m(\boldsymbol{x}, t_2)$, then $\alpha = \tilde{a}$, $\beta = \tilde{b}$ and the matrix element (7) could be considered by the usual techniques, based on the reduction of the S matrix to the normal form relative to the vacuum $|0\rangle_{\text{in}}$. However, as has been shown in appendix 1, the spinors $\{{}^{\pm}\phi_m(\boldsymbol{x})\}$ must satisfy condition (A.4). It is not difficult to guess that any system of spinors $\{{}_{\pm}\psi^0_m(\boldsymbol{x})\}$ could be chosen deliberately to satisfy the requirements of (A.4). Let us choose $\{{}_{\pm}\phi_m(\boldsymbol{x})\}$ as such a system. Then, for the system ${}^{\pm}\phi_m(\boldsymbol{x}) = {}_{\pm}\phi_m(\boldsymbol{x}, t_2)$, condition (A.4) has the form of (A.8) and is the condition for the finiteness of the total number of pairs created by the field. Obviously, condition (A.8) is not realised if, at $\mathscr{V} \to \infty$, the total number of particles created approaches infinity (this must be the case for the model of an external field having an infinite energy in the whole space). Consequently, for the field which creates pairs in the whole space, it is not possible to make such a choice for the final states (i.e. so that $\alpha = \tilde{a}$, $\beta = \tilde{b}$). This means that the usual techniques must be generalised.

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